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ON SOME NEW TENSORS AND THEIR PROPERTIES IN A FIVE-DIMENSIONAL FINSLER SPACE-III

S. C. Rastogi

Professor, Department of Mathematics, Seth Vishambhar Nath Institute of Engineering Research and Technology (SVNIERT), Barabanki, Uttar Pradesh, India

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ABSTRACT

Berwald [1, 2] developed the study of two-dimensional Finsler spaces, whose idea was followed by Moor [9] to introduced in a three-dimensional Finsler space the intrinsic field of orthonormal frame consisting of normalized support element l^i, normalized torsion vector m^i and the unit vector n^i, orthogonal to both l^i and m^i. Various aspects of three-dimensional Finsler spaces have been studied by Rund [5], Matsumoto [6,7,8], Rastogi [12,13,14] and others. Similarly four-dimensional Finsler spaces have been studied by Pandey and Dwivedi [10] and Rastogi [15] etc. Theory of five-dimensional Finsler spaces in terms of scalars has been studied by Pandey, Dwivedi and Gupta [11] and Dwivedi, Rastogi and Dwivedi [4]. In 1990, certain new tensors were defined and studied by Rastogi [12], while in 2019 Rastogi [14] introduced a new tensor D_{ijk} in three-dimensional Finsler space, which is similar to tensor C_{ijk}, but satisfies different properties like $D_{ijk}\, l^i = 0$ and $D_{ijk}g^{jk} = D_i = D\, n_i$. This tensor exists only in Finsler spaces of more than two-dimensions. This tensor was further studied in four-dimensional Finsler space by Rastogi [15], but it is important to note that there are two tensors of such type in four-dimensional Finsler space. In this paper besides studying variety of tensors and their properties in a five-dimensional Finsler space, we have also studied various kinds of D-tensors which are actually three in F^5.

KEYWORDS: *Five-Dimensional Finsler Spaces, D-Tensors, Q-Tensor, D-Reducibility*

INTRODUCTION

Let F^5 be a five-dimensional Finsler space equipped with a fundamental function L(x,y), orthonormal Miron frame $e_{\alpha)I}$ (α = 1,2,3,4,5), adopted components of the metric tensor g_{ij} and E-tensors ε_{ijklm} respectively given by $\delta_{\alpha\beta}$ and $\varepsilon_{\alpha\beta\gamma\delta\Theta} = (\delta^{1\ 2\ 3\ 4\ 5}_{\alpha\ \beta\ \gamma\ \delta\ \Theta})$, where right hand term is generalised Kronecker delta and satisfies usual properties [11, 15]. In a five-dimensional Finsler space we have five orthonormal unit vectors, which shall be denoted by l_i, m_i, $n_{(1)I}$, $n_{(2)I}$ and $n_{(3)i}$. The h-covariant derivative $e_{\alpha)}{}^i{}_{/j}$ of the vector $e_{\alpha)}{}^I$, is given as

$$e_{1)}{}^i{}_{/j} = l^i{}_{/j} = 0,\ e_{2)}{}^i{}_{/j} = m^i{}_{/j} = n_{(1)}{}^I\, h_{(1)j} - n_{(2)}{}^I\, h_{(3)j} - n_{(3)}{}^I\, h_{(4)j},\ e_{3)}{}^i{}_{/j} = n_{(1)}{}^i{}_{/j} = n_{(2)}{}^I\, h_{(2)j} - m^i\, h_{(1)j} - n_{(3)}{}^I\, h_{(5)j},$$

$$e_{4)}{}^i{}_{/j} = n_{(2)}{}^i{}_{/j} = m^i\, h_{(3)j} - n_{(1)}{}^I\, h_{(2)j} - n_{(3)}{}^I\, h_{(6)j},\ e_{5)}{}^i{}_{/j} = n_{(3)}{}^i{}_{/j} = m^i\, h_{(4)j} + n_{(1)}{}^I\, h_{(5)j} + n_{(2)}{}^I\, h_{(6)j} \qquad (1.1)$$

where $h_{(1)j}$, $h_{(2)j}$, $h_{(3)j}$, $h_{(4)j}$, $h_{(5)j}$ and $h_{(6)j}$ are called h-connection vectors of F^5.

The v-covariant derivative $e_{\alpha)}{}^i{}_{//j}$ of the vector $e_{\alpha)}{}^I$ is expressed as

$$e_{1)}{}^i{}_{//j} = l^i{}_{//j} = L^{-1}h^i_j = L^{-1}(m^i\, m_j + n_{(1)}{}^I\, n_{(1)j} + n_{(2)}{}^I\, n_{(2)j} + n_{(3)}{}^I\, n_{(3)j}),$$

$e_{(2)}{}^{i}{}_{//j} = m^{i}{}_{//j} = L^{-1}(-l^{i} m_{j} + n_{(1)}{}^{I} U_{(1)j} + n_{(2)}{}^{I} U_{(2)j} + n_{(3)}{}^{I} U_{(4)j}),$

$e_{(3)}{}^{i}{}_{//j} = n_{(1)}{}^{i}{}_{//j} = L^{-1}(-l^{i} n_{(1)j} - m^{i} U_{(1)j} + n_{(2)}{}^{I} U_{(3)j} + n_{(3)}{}^{I} U_{(5)j}),$

$e_{(4)}{}^{i}{}_{//j} = n_{(2)}{}^{i}{}_{//j} = L^{-1}(-l^{i} n_{(2)j} - m^{i} U_{(2)j} - n_{(1)}{}^{I} U_{(3)j} + n_{(3)}{}^{I} U_{(6)j}),$

$e_{(5)}{}^{i}{}_{//j} = n_{(3)}{}^{i}{}_{//j} = L^{-1}(-l^{i} n_{(3)j} - m^{i} U_{(4)j} - n_{(1)}{}^{I} U_{(5)j} - n_{(2)}{}^{I} U_{(6)j}),$ (1.2)

where $U_{(1)j}$, $U_{(2)j}$, $U_{(3)j}$, $U_{(4)j}$, $U_{(5)j}$ and $U_{(6)j}$ are called v-connection vectors.

Cartan's tensor [3],Cijk in F5 can be expressed as

$L\, C_{ijk} = C_{(1)} m_{i} m_{j} m_{k} + C_{(2)} n_{(1)I} n_{(1)j} n_{(1)k} + C_{(3)} n_{(2)i} n_{(2)j} n_{(3)k} + C_{(4)} n_{(3)I} n_{(3)j} n_{(3)k}$

$+ \sum_{(I,j,k)}[C_{(5)} m_{i} m_{j} n_{(1)k} + C_{(6)} m_{i} m_{j} n_{(2)k} + C_{(7)} m_{i} m_{j} n_{(3)k} + C_{(8)} n_{(1)I} n_{(1)j} m_{k}$

$+ C_{(9)} n_{(1)i} n_{(1)j} n_{(2)k} + C_{(10)} n_{(1)I} n_{(1)j} n_{(3)k} + C_{(11)} n_{(2)I} n_{(2)j} m_{k} + C_{(12)} n_{(2)I} n_{(2)j} n_{(1)k}$

$+ C_{(13)} n_{(2)I} n_{(2)j} n_{(3)k} + C_{(14)} n_{(3)I} n_{(3)j} m_{k} + C_{(15)} n_{(3)I} n_{(3)j} n_{(1)k} + C_{(16)} n_{(3)I} n_{(3)j} n_{(2)k}$

$+ C_{(17)} m_{i}(n_{(1)j} n_{(2)k} + n_{(1)k} n_{(2)j}) + C_{(18)} m_{i}(n_{(1)j} n_{(3)k} + n_{(1)k} n_{(3)j})$

$+ C_{(19)} m_{i}(n_{(2)j} n_{(3)k} + n_{(2)k} n_{(3)j}) + C_{(20)} n_{(1)i}(n_{(2)j} n_{(3)k} + n_{(2)k} n_{(3)j})$ (1.3)

Where

$C_{(1)} + C_{(8)} + C_{(11)} + C_{(14)} = L\, C,\ C_{(2)} + C_{(5)} + C_{(12)} + C_{(15)} = 0,$

$C_{(3)} + C_{(6)} + C_{(9)} + C_{(16)} = 0,\ C_{(4)} + C_{(7)} + C_{(10)} + C_{(13)} = 0$ (1.4)

and $C_{(17)}$,$C_{(18)}$, $C_{(19)}$ and $C_{(20)}$ are non-zero scalars in F^{5}.

SECOND ORDER TENSORS AND THEIR h-COVARIANT DERIVATIVES

Definition 2.1: In a Finsler space of five-dimensions F^{5}, we define following ten non-zero second order symmetric tensors.

${}^{1}A_{ij}(x,y) = \sum_{(ij)}\{l_{i} m_{j}\}$, ${}^{2}A_{ij}(x,y) = \sum_{(ij)}\{l_{i} n_{(1)j}\}$, ${}^{3}A_{ij}(x,y) = \sum_{(ij)}\{l_{i} n_{(2)j}\}$, ${}^{4}A_{ij}(x,y) = \sum_{(ij)}\{l_{i} n_{(3)j}\}$, (2.1)a

${}^{5}A_{ij}(x,y) = \sum_{(ij)}\{m_{i} n_{(1)j}\}$, ${}^{6}A_{ij}(x,y) = \sum_{(ij)}\{m_{i} n_{(2)j}\}$, ${}^{7}A_{ij}(x,y) = \sum_{(ij)}\{m_{i} n_{(3)j}\}$, (2.1)b

${}^{8}A_{ij}(x,y) = \sum_{(ij)}\{n_{(1)i} n_{(2)j}\}$, ${}^{9}A_{ij}(x,y) = \sum_{(ij)}\{n_{(1)i} n_{(3)j}\}$, ${}^{10}A_{ij}(x,y) = \sum_{(ij)}\{n_{(2)i} n_{(3)j}\}$. (2.1)c

From equations (2.1)a,b,c, by virtue of equation (1.1), we can obtain

${}^{1}A_{ij/k} = h_{(1)k}\,{}^{2}A_{ij} - h_{(3)k}\,{}^{3}A_{ij} - h_{(4)k}\,{}^{4}A_{ij}$, ${}^{2}A_{ij/k} = h_{(2)k}\,{}^{3}A_{ij} - h_{(1)k}\,{}^{1}A_{ij} - h_{(5)k}\,{}^{4}A_{ij}$, (2.2)a

${}^{3}A_{ij/k} = h_{(3)k}\,{}^{1}A_{ij} - h_{(2)k}\,{}^{2}A_{ij} - h_{(6)k}\,{}^{4}A_{ij}$, ${}^{4}A_{ij/k} = h_{(4)k}\,{}^{1}A_{ij} + h_{(5)k}\,{}^{2}A_{ij} + h_{(6)k}\,{}^{3}A_{ij}$, (2.2)b

${}^{5}A_{ij/k} = 2\, h_{(1)k}(n_{(1)I} n_{(1)j} - m_{i} m_{j}) + h_{(2)k}\,{}^{6}A_{ij} - h_{(3)k}\,{}^{8}A_{ij} - h_{(4)k}\,{}^{9}A_{ij} - h_{(5)k}\,{}^{7}A_{ij}$, (2.2)c

${}^{6}A_{ij/k} = h_{(1)k}\,{}^{8}A_{ij} - h_{(2)k}\,{}^{5}A_{ij} + 2\, h_{(3)k}(m_{i} m_{j} - n_{(2)I} n_{(2)j}) - h_{(4)k}\,{}^{10}A_{ij} - h_{(6)k}\,{}^{7}A_{ij}$, (2.2)d

${}^{7}A_{ij/k} = h_{(1)k}\,{}^{9}A_{ij} - h_{(3)k}\,{}^{10}A_{ij} + 2\, h_{(4)k}(m_{i} m_{j} - n_{(30i} n_{(3)j}) + h_{(5)k}\,{}^{5}A_{ij} + h_{(6)k}\,{}^{6}A_{ij}$, (2.2)e

${}^{8}A_{ij/k} = -\, h_{(1)k}\,{}^{6}A_{ij} + 2\, h_{(2)k}(n_{(2)I} n_{(2)j} - n_{(1)I} n_{(1)j}) + h_{(3)k}\,{}^{5}A_{ij} - h_{(5)k}\,{}^{10}A_{ij} - h_{(6)k}\,{}^{9}A_{ij}$, (2.2)f

$^{9}A_{ij/k} = - h_{(1)k}{}^{7}A_{ij} + h_{(2)k}{}^{10}A_{ij} + h_{(4)k}{}^{5}A_{ij} + 2h_{(5)k}(n_{(1)I}\, n_{(1)j} – n_{(3)I}\, n_{(3)j}) + h_{(6)k}{}^{8}A_{ij},$ (2.2)g

$^{10}A_{ij/k} = - h_{(2)k}{}^{9}A_{ij} + h_{(3)k}{}^{7}A_{ij} + h_{(4)k}{}^{6}A_{ij} + h_{(5)k}{}^{8}A_{ij} + 2\, h_{(6)k}\, (n_{(2)I}\, n_{(2)j} – n_{(3)I}\, n_{(3)j}).$ (2.2)h

From equations (2.2) a,b,c,d,e,f,g,h, we can obtain

Theorem 2.1: In a five-dimensional Finsler space F^5, tensors $^{1}A_{ij/k}$, $^{2}A_{ij/k}$, $^{3}A_{ij/k}$and $^{4}A_{ij/k}$ satisfy equation

$$^{1}A_{ij/k} + {}^{2}A_{ij/k} + {}^{3}A_{ij/k} + {}^{4}A_{ij/k} = (h_{(3)k} + h_{(4)k} – h_{(1)k})\, {}^{1}A_{ij} + (h_{(1)k} – h_{(2)k} + h_{(5)k})\, {}^{2}A_{ij} + (h_{(2)k} – h_{(3)k} + h_{(6)k})\, {}^{3}A_{ij} – (h_{(4)k} + h_{(5)k} + h_{(6)k})\, {}^{4}A_{ij} \quad (2.3)$$

Theorem 2.2: In a five-dimensional Finsler space F^5, tensors $^{5}A_{ij/k}$, $^{6}A_{ij/k}$ and $^{7}A_{ij/k}$ satisfy equation

$$^{5}A_{ij/k} + {}^{6}A_{ij/k} + {}^{7}A_{ij/k} = (h_{(5)k} – h_{(2)k})\, {}^{5}A_{ij} + (h_{(2)k} + h_{(6)k})\, {}^{6}A_{ij} – (h_{(5)k} + h_{(6)k})\, {}^{7}A_{ij} + (h_{(1)k} – h_{(3)k})\, {}^{8}A_{ij} + (h_{(1)k} – h_{(4)k})\, {}^{9}A_{ij} – (h_{(3)k} + h_{(4)k})\, {}^{10}A_{ij} + 2(h_{(3)k} + h_{(4)k} – h_{(1)k})m_i\, m_j + 2(h_{(1)k}\, n_{(1)I}\, n_{(1)j} – h_{(3)k}\, n_{(2)I}\, n_{(2)j} – h_{(4)k}\, n_{(3)I}\, n_{(3)j}) \quad (2.4)$$

Theorem 2.3: In a five-dimensional Finsler space F^5, tensors $^{8}A_{ij/k}$, $^{9}A_{ij/k}$ and $^{10}A_{ij/k}$ satisfy equation

$$^{8}A_{ij/k} + {}^{9}A_{ij/k} + {}^{10}A_{ij/k} = (h_{(3)k} + h_{(4)k})\, {}^{5}A_{ij} + (h_{(4)k} – h_{(1)k})\, {}^{6}A_{ij} + (h_{(3)k} – h_{(1)k})\, {}^{7}A_{ij} + (h_{(5)k} + h_{(6)k})({}^{8}A_{ij} – 2\, n_{(3)I}\, n_{(3)j}) – (h_{(2)k} + h_{(6)k})({}^{9}A_{ij} – 2\, n_{(2)I}\, n_{(2)j}) + (h_{(2)k} – h_{(5)k})({}^{10}A – 2\, n_{(1)I}\, n_{(10j)}) \quad (2.5)$$

Definition 2.2: In a five-dimensional Finsler space F^5, we define following symmetric tensors

$^{1}B_{ij} = m_i\, m_j$, $^{2}B_{ij} = n_{(1)I}\, n_{(1)j}$, $^{3}B_{in} = n_{(2)I}\, n_{(2)j}$ and $^{4}B_{ij} = n_{(3)I}\, n_{(3)j}$ (2.6)

From equation (2.6), we can obtain

$^{1}B_{ij/k} = h_{(1)k}{}^{5}A_{ij} – h_{(3)k}{}^{6}A_{ij} – h_{(4)k}{}^{7}A_{ij}$, $^{2}B_{ij/k} = - h_{(1)k}{}^{5}A_{ij} + h_{(2)k}{}^{8}A_{ij} – h_{(5)k}{}^{9}A_{ij}$, (2.7)a

$^{3}B_{ij/k} = - h_{(2)k}{}^{8}A_{ij} + h_{(3)k}{}^{6}A_{ij} – h_{(6)k}{}^{10}A_{ij}$, $^{4}B_{ij/k} = h_{(4)k}{}^{7}A_{ij} + h_{(5)k}{}^{9}A_{ij} + h_{(6)k}{}^{10}A_{ij}$ (2.7)b

which lead to

Theorem 2.4: In a five-dimensional Finsler space F^5, equation (2.7)a,b lead to

$$^{1}B_{ij/k} + {}^{2}B_{ij/k} + {}^{3}B_{ij/k} + {}^{4}B_{ij/k} = 0. \quad (2.8)$$

Remark. Theorem2.4: is actually representing that h-covariant derivative of angular metric tensor in a five-dimensional Finsler space vanishes.

Definition 2.3: In a five-dimensional Finsler space F^5, we define following symmetric tensors.

$^{1}T_{ij} = m_i\, m_j + n_{(1)I}\, n_{(1)j}$, $^{2}T_{ij} = m_i\, m_j + n_{(2)I}\, n_{(2)j}$, $^{3}T_{ij} = m_i\, m_j + n_{(3)I}\, n_{(3)j}$, (2.9)a

$^{4}T_{ij} = n_{(1)I}\, n_{(1)j} + n_{(2)I}\, n_{(2)j}$, $^{5}T_{ij} = n_{(1)I}\, n_{(1)j} + n_{(3)I}\, n_{(3)j}$, $^{6}T_{ij} = n_{(2)I}\, n_{(2)j} + n_{(3)I}\, n_{(3)j}$ (2.9)b

From equation (2.9)a,b, we can obtain

$^{1}T_{ij/k} = h_{(2)k}{}^{8}A_{ij} – h_{(3)k}{}^{6}A_{ij} – h_{(4)k}{}^{7}A_{ij} – h_{(5)k}{}^{9}A_{ij}$, (2.10)a

$^{2}T_{ij/k} = h_{(1)k}\,^{5}A_{ij} - h_{(2)k}\,^{8}A_{ij} - h_{(4)k}\,^{7}A_{ij} - h_{(6)k}\,^{10}A_{ij},$ (2.10)b

$^{3}T_{ij/k} = h_{(1)k}\,^{5}A_{ij} - h_{(3)k}\,^{6}A_{ij} + h_{(5)k}\,^{9}A_{ij} + h_{(6)k}\,^{10}A_{ij}$ (2.10)c

If we find h-covariant derivative of remaining three terms, we can obtain

Theorem 2.5: In a five-dimensional Finsler space F^5, tensors defined in equations (2.9)a,b satisfy equation

$^{1}T_{ij/k} + {}^{6}T_{ij/k} = 0,\ {}^{2}T_{ij/k} + {}^{5}T_{ij/k} = 0$ and $^{3}T_{ij/k} + {}^{4}T_{ij/k} = 0.$ (2.11)

Definition 2.4: In a five-dimensional Finsler space F^5, we define following symmetric tensors.

$^{1}U_{ij} = m_i\, m_j - n_{(1)I}\, n_{(1)j},\ {}^{2}U_{ij} = m_i\, m_j - n_{(2)I}\, n_{(2)j},\ {}^{3}U_{ij} = m_i\, m_j - n_{(3)I}\, n_{(3)j},$ (2.12)a

$^{4}U_{ij} = n_{(1)I}\, n_{(1)j} - n_{(2)I}\, n_{(2)j},\ {}^{5}U_{ij} = n_{(1)I}\, n_{(1)j} - n_{(3)I}\, n_{(3)j},\ {}^{6}U_{ij} = n_{(2)I}\, n_{(2)j} - n_{(3)I}\, n_{(3)j}$ (2.12)b

From equation (2.12)a,b, we can easily obtain

$^{1}U_{ij/k} = 2\, h_{(1)k}\,^{5}A_{ij} - h_{(2)k}\,^{8}A_{ij} - h_{(3)k}\,^{6}A_{ij} - h_{(4)k}\,^{7}A_{ij} + h_{(5)k}\,^{9}A_{ij},$ (2.13)a

$^{2}U_{ij/k} = h_{(1)k}\,^{5}A_{ij} + h_{(2)k}\,^{8}A_{ij} - 2\, h_{(3)k}\,^{6}A_{ij} - h_{(4)k}\,^{7}A_{ij} + h_{(6)k}\,^{10}A_{ij},$ (2.13)b

$^{3}U_{ij/k} = h_{(1)k}\,^{5}A_{ij} - h_{(3)k}\,^{6}A_{ij} - 2h_{(4)k}\,^{7}A_{ij} - h_{(5)k}\,^{9}A_{ij} - h_{(6)k}\,^{10}A_{ij},$ (2.13)c

$^{4}U_{ij/k} = - h_{(1)k}\,^{5}A_{ij} + 2\, h_{(2)k}\,^{8}A_{ij} - h_{(3)k}\,^{6}A_{ij} - h_{(5)k}\,^{9}A_{ij} + h_{(6)k}\,^{10}A_{ij},$ (2.13)d

$^{5}U_{ij/k} = - h_{(1)k}\,^{5}A_{ij} + h_{(2)k}\,^{8}A_{ij} - h_{(4)k}\,^{7}A_{ij} - 2\, h_{(5)k}\,^{9}A_{ij} - h_{(6)k}\,^{10}A_{ij},$ (2.13)e

$^{6}U_{ij/k} = - h_{(2)k}\,^{8}A_{ij} + h_{(3)k}\,^{6}A_{ij} - h_{(4)k}\,^{7}A_{ij} - h_{(5)k}\,^{9}A_{ij} - 2\, h_{(6)k}\,^{10}A_{ij}.$ (2.13)f

These equations in (2.13)a,b,c,d,e,f, lead us to

$^{1}U_{ij/k} + {}^{5}U_{ij/k} = {}^{3}U_{ij/k}, {}^{2}U_{ij/k} + {}^{6}U_{ij/k} = {}^{3}U_{ij/k}$ (2.14)a

and

$^{3}U_{ij/k} + {}^{4}U_{ij/k} = 2(h_{(2)k}\,^{8}A_{ij} - h_{(3)k}\,^{6}A_{ij} - h_{(4)k}\,^{7}A_{ij} - h_{(5)k}\,^{9}A_{ij})$ (2.14)b

Hence:

Theorem 2.6: In a five-dimensional Finsler space F^5, tensors $U_{ij/k}$ satisfy equations (2.14)a,b in the following form:

$^{1}E_{ij} = l_i\, m_j - l_j\, m_i,\ {}^{2}E_{ij} = l_i n_{(1)j} - l_j\, n_{(1)I},\ {}^{3}E_{ij} = l_i\, n_{(2)j} - l_j\, n_{(2)I},\ {}^{4}E_{ij} = l_i\, n_{(3)j} - l_j\, n_{(3)I},$ (2.15)a

$^{5}E_{ij} = m_i n_{(1)j} - m_j\, n_{(1)I},\ {}^{6}E_{ij} = m_i\, n_{(2)j} - m_j\, n_{(2)I},\ {}^{7}E_{ij} = m_i\, n_{(3)j} - m_j\, n_{(3)I},$ (2.15)b

$^{8}E_{ij} = n_{(1)I}\, n_{(2)j} - n_{(1)j}\, n_{(2)I},\ {}^{9}E_{ij} = n_{(1)I}\, n_{(3)j} - n_{(1)j}\, n_{(3)I},\ {}^{10}E_{ij} = n_{(2)I}\, n_{(3)j} - n_{(2)j}\, n_{(3)i}.$ (2.15)c

From equations (2.15) a, b, c, we can obtain on simplification

$^{1}E_{ij/k} = h_{(1)k}\,^{2}E_{ij} - h_{(3)k}\,^{3}E_{ij} - h_{(4)k}\,^{4}E_{ij},\ {}^{2}E_{ij/k} = - h_{(1)k}\,^{1}E_{ij} + h_{(2)k}\,^{3}E_{ij} - h_{(5)k}\,^{4}E_{ij},$ (2.16)a

$^{3}E_{ij/k} = - h_{(2)k}\,^{2}E_{ij} + h_{(3)k}\,^{1}E_{ij} - h_{(6)k}\,^{4}E_{ij},\ {}^{4}E_{ij/k} = h_{(4)k}\,^{1}E_{ij} + h_{(5)k}\,^{2}E_{ij} + h_{(6)k}\,^{3}E_{ij},$ (2.16)b

$$^{5}E_{ij/k} = h_{(2)k}{}^{6}E_{ij} + h_{(3)k}{}^{8}E_{ij} + h_{(4)k}{}^{9}E_{ij} - h_{(5)k}{}^{7}E_{ij}, \quad (2.16)c$$

$$^{6}E_{ij/k} = h_{(1)k}{}^{8}E_{ij} - h_{(2)k}{}^{5}E_{ij} + h_{(4)k}{}^{10}E_{ij} - h_{(6)k}{}^{7}E_{ij}, \quad (2.16)d$$

$$^{7}E_{ij/k} = h_{(1)k}{}^{9}E_{ij} - h_{(3)k}{}^{10}E_{ij} + h_{(5)k}{}^{5}E_{ij} + h_{(6)k}{}^{6}E_{ij}, \quad (2.16)e$$

$$^{8}E_{ij/k} = - h_{(1)k}{}^{6}E_{ij} - h_{(3)k}{}^{5}E_{ij} + h_{(5)k}{}^{10}E_{ij} - h_{(6)k}{}^{9}E_{ij}, \quad (2.16)f$$

$$^{9}E_{ij/k} = - h_{(1)k}{}^{7}E_{ij} + h_{(2)k}{}^{10}E_{ij} - h_{(4)k}{}^{5}E_{ij} + h_{(6)k}{}^{8}E_{ij}, \quad (2.16)g$$

$$^{10}E_{ij/k} = - h_{(2)k}{}^{9}E_{ij} + h_{(3)k}{}^{7}E_{ij} - h_{(4)k}{}^{6}E_{ij} - h_{(5)k}{}^{8}E_{ij} \quad (2.16)h$$

From these equations we can obtain

$$^{1}E_{ij/k} + {}^{2}E_{ij/k} + {}^{3}E_{ij/k} + {}^{4}E_{ij/k} = {}^{1}E_{ij}(h_{(3)k} + h_{(4)k} - h_{(1)k}) + {}^{2}E_{ij}(h_{(1)k} + h_{(5)k} - h_{(2)k})$$
$$+ {}^{3}E_{ij}(h_{(2)k} + h_{(6)k} - h_{(3)k}) - {}^{4}E_{ij}(h_{(4)k} + h_{(5)k} + h_{(6)k}) \quad (2.17)$$

and

$$^{5}E_{ij/k} + {}^{6}E_{ij/k} + {}^{7}E_{ij/k} + {}^{8}E_{ij/k} + {}^{9}E_{ij/k} + {}^{10}E_{ij/k}$$
$$= {}^{5}E_{ij}(h_{(5)k} - h_{(2)k} - h_{(3)k} - h_{(4)k}) + {}^{6}E_{ij}(h_{(2)k} + h_{(6)k} - h_{(1)k} - h_{(4)k}) + {}^{7}E_{ij}(h_{(3)k} - h_{(1)k} - h_{(5)k} - h_{(6)k})$$
$$+ {}^{8}E_{ij}(h_{(1)k} + h_{(3)k} + h_{(6)k} - h_{(5)k}) + {}^{9}E_{ij}(h_{(1)k} + h_{(4)k} - h_{(2)k} - h_{(6)k})$$
$$+ {}^{10}E_{ij}(h_{(2)k} + h_{(4)k} + h_{(5)k} - h_{(3)k}) \quad (2.18)$$

Hence:

Theorem 2.7: In a five-dimensional Finsler space F^5, h-covariant derivatives of skew-symmetric tensors given by equations (2.15)a,b,c satisfy equations (2.17) and (2.18).

V-COVARIANT DERIVATIVES OF TENSORS DEFINED ABOVE

For the terms defined in equation (2.1), with the help of definition (1.2) of v-covariant

$$^{1}A_{ij//k} = L^{-1}(h_{ik}\, m_j + h_{jk}\, m_i - 2l_il_jm_k + U_{(1)k}{}^{2}A_{ij} + U_{(2)k}{}^{3}A_{ij} + U_{(4)k}{}^{4}A_{ij}), \quad (3.1)a$$

$$^{2}A_{ij//k} = L^{-1}(h_{ik}n_{(1)j} + h_{jk}\, n_{(1)I} - 2\, l_il_j\, n_{(1)k} - U_{(1)k}{}^{1}A_{ij} + U_{(3)k}{}^{3}A_{ij} + U_{(5)k}{}^{4}A_{ij}), \quad (3.1)b$$

$$^{3}A_{ij//k} = L^{-1}(h_{ik}n_{(2)j} + h_{jk}\, n_{(2)I} - 2\, l_il_j\, n_{(2)k} - U_{(2)k}{}^{1}A_{ij} - U_{(3)k}{}^{2}A_{ij} + U_{(6)k}{}^{4}A_{ij}), \quad (3.1)c$$

$$^{4}A_{ij//k} = L^{-1}(h_{ik}n_{(3)j} + h_{jk}\, n_{(3)I} - 2\, l_il_j\, n_{(3)k} - U_{(4)k}{}^{1}A_{ij} - U_{(5)k}{}^{2}A_{ij} - U_{(6)k}{}^{3}A_{ij}). \quad (3.1)d$$

Similarly, from equations of (2.1) b, c we get

$$^{5}A_{ij//k} = L^{-1}\{U_{(2)k}{}^{8}A_{ij} + U_{(3)k}{}^{6}A_{ij} + U_{(4)k}{}^{9}A_{ij} + U_{(5)k}{}^{7}A_{ij}$$
$$+ 2\, U_{(1)k}(n_{(1)I}\, n_{(1)j} - m_i\, m_j) - m_k{}^{2}A_{ij} - n_{(1)k}{}^{1}A_{ij}\}, \quad (3.2)a$$

$$^{6}A_{ij//k} = L^{-1}\{U_{(1)k}{}^{8}A_{ij} - U_{(3)k}{}^{5}A_{ij} + U_{(4)k}{}^{10}A_{ij} + U_{(6)k}{}^{7}A_{ij}$$
$$+ 2\, U_{(2)k}(n_{(2)I}\, n_{(2)j} - m_i\, m_j) - m_k{}^{3}A_{ij} - n_{(2)k}{}^{1}A_{ij}\}, \quad (3.2)b$$

$$^{7}A_{ij//k} = L^{-1}\{U_{(1)k}{}^{7}A_{ij} + U_{(2)k}{}^{10}A_{ij} - U_{(5)k}{}^{5}A_{ij} - U_{(6)k}{}^{6}A_{ij}$$

$$+ 2\, U_{(4)k}(n_{(3)I}\, n_{(3)j} - m_i\, m_j) - m_k\, {}^4A_{ij} - n_{(3)k}\, {}^1A_{ij}\}, \quad (3.2)c$$

$${}^8A_{ij//k} = L^{-1}\{-U_{(1)k}\, {}^6A_{ij} - U_{(2)k}\, {}^5A_{ij} + U_{(5)k}\, {}^{10}A_{ij} + U_{(6)k}\, {}^9A_{ij}$$

$$+ 2\, U_{(3)k}(n_{(2)I}\, n_{(2)j} - n_{(1)I}\, n_{(1)j}) - n_{(1)k}\, {}^3A_{ij} - n_{(2)k}\, {}^2A_{ij}\}, \quad (3.2)d$$

$${}^9A_{ij//k} = L^{-1}\{-U_{(1)k}\, {}^7A_{ij} + U_{(3)k}\, {}^{10}A_{ij} - U_{(4)k}\, {}^5A_{ij} - U_{(6)k}\, {}^8A_{ij}$$

$$+ 2\, U_{(5)k}(n_{(3)I}\, n_{(3)j} - n_{(1)I}\, n_{(1)j}) - n_{(1)k}\, {}^4A_{ij} - n_{(3)k}\, {}^2A_{ij}\}, \quad (3.2)e$$

$${}^{10}A_{ij//k} = L^{-1}\{-U_{(2)k}{}^7A_{ij} - U_{(3)k}{}^9A_{ij} - U_{(4)k}\, {}^6A_{ij} - U_{(5)k}\, {}^8A_{ij}$$

$$+ 2\, U_{(6)k}(n_{(3)I}\, n_{(3)j} - n_{(2)I}\, n_{(2)j}) - n_{(2)k}\, {}^4A_{ij} - n_{(3)k}{}^3A_{ij}\}. \quad (3.2)f$$

For tensors defined by equation (2.6), we can obtain

$${}^1B_{ij//k} = L^{-1}(-\, m_k{}^1A_{ij} + U_{(1)k}{}^5A_{ij} + U_{(2)k}{}^6A_{ij} + U_{(4)k}{}^7A_{ij}), \quad (3.3)a$$

$${}^2B_{ij//k} = L^{-1}(-\, n_{(1)k}{}^2A_{ij} - U_{(1)k}{}^5A_{ij} + U_{(3)k}{}^8A_{ij} + U_{(5)k}{}^9A_{ij}), \quad (3.3)b$$

$${}^3B_{ij//k} = L^{-1}(-\, n_{(2)k}{}^3A_{ij} - U_{(2)k}{}^6A_{ij} - U_{(3)k}{}^8A_{ij} + U_{(6)k}{}^{10}A_{ij}), \quad (3.3)c$$

$${}^4B_{ij//k} = L^{-1}(-\, n_{(3)k}{}^4A_{ij} - U_{(4)k}{}^7A_{ij} - U_{(5)k}{}^9A_{ij} - u_{(6)k}{}^{10}A_{ij}). \quad (3.3)d$$

From equations (3.3) a,b,c,d, we can obtain

Theorem 3.1: In a five-dimensional Finsler space F^5, tensors given in (3.3) satisfy equation

$${}^1B_{ij//k} + {}^2B_{ij//k} + {}^3B_{ij//k} + {}^4B_{ij//k} = -\, L^{-1}(m_k{}^1A_{ij} + n_{(1)k}{}^2A_{ij} + n_{(2)k}{}^3A_{ij} + n_{(3)k}{}^4A_{ij}) \quad (3.4)$$

From equations (2.9) a,b we can obtain

$${}^1T_{ij//k} = L^{-1}[-m_k{}^1A_{ij} - n_{(1)k}{}^2A_{ij} + U_{(2)k}{}^6A_{ij} + U_{(3)k}{}^8A_{ij} + U_{(4)k}{}^7A_{ij} + U_{(5)k}{}^9A_{ij}], \quad (3.5)a$$

$${}^2T_{ij//k} = L^{-1}[-\, m_k\, {}^1A_{ij} - n_{(2)k}{}^3A_{ij} + U_{(1)k}\, {}^5A_{ij} - U_{(3)k}{}^8A_{ij} + U_{(4)k}{}^7A_{ij} + U_{(6)k}{}^{10}A_{ij}], \quad (3.5)b$$

$${}^3T_{ij//k} = L^{-1}[-m_k{}^1A_{ij} - n_{(3)k}{}^4A_{ij} + U_{(1)k}{}^5A_{ij} + U_{(2)k}\, {}^6A_{ij} - U_{(5)k}{}^9A_{ij} - U_{(6)k}{}^{10}A_{ij}], \quad (3.5)c$$

$${}^4T_{ij//k} = L^{-1}[-n_{(1)k}{}^2A_{ij} - n_{(2)k}\, {}^3A_{ij} - U_{(1)k}{}^5A_{ij} - U_{(2)k}{}^6A_{ij} + U_{(5)k}{}^9A_{ij} + U_{(6)k}{}^{10}A_{ij}], \quad (3.5)d$$

$${}^5T_{ij//k} = L^{-1}[-n_{(1)k}{}^2A_{ij} - n_{(3)k}\, {}^4A_{ij} - U_{(1)k}{}^5A_{ij} + U_{(3)k}{}^8A_{ij} - U_{(4)k}{}^7A_{ij} - U_{(6)k}{}^{10}A_{ij}], \quad (3.5)e$$

$${}^6T_{ij//k} = L^{-1}[-n_{(2)k}{}^3A_{ij} - n_{(3)k}{}^4A_{ij} - U_{(2)k}{}^6A_{ij} - U_{(3)k}{}^8A_{ij} - U_{(4)k}{}^7A_{ij} - U_{(5)k}{}^9A_{ij}]. \quad (3.5)f$$

Hence:

Theorem 3.2: In a five-dimensional Finsler space F^5, tensors given in (2.9)a,b satisfy equations (3.5)a,b,c,d,e,f.

From equation (3.5) a,b,c,d,e,f, we can further obtain

$${}^1T_{ij//k} + {}^6T_{ij//k} = {}^2T_{ij//k} + {}^5T_{ij//k} = {}^3T_{ij//k} + {}^4T_{ij//k} = L^{-1}[-m_k{}^1A_{ij} - n_{(1)k}{}^2A_{ij} - n_{(2)k}{}^3A_{ij} - n_{(3)k}{}^4A_{ij}] \quad (3.6).$$

Hence:

Theorem 3.3: In a five-dimensional Finsler space F^5, tensors given in (3.5) a,b,c,d,e,f, satisfy equation (3.6).

From equation (2.12) a,b, we can obtain

$${}^{1}U_{ij//k} = L^{-1}[-m_k\, {}^{1}A_{ij} + n_{(1)k}{}^{2}A_{ij} + 2\, U_{(1)k}{}^{5}A_{ij} + U_{(2)k}{}^{6}A_{ij} - U_{(3)k}{}^{8}A_{ij} + U_{(4)k}{}^{7}A_{ij} - U_{(5)k}{}^{9}A_{ij}], \quad (3.7)a$$

$${}^{2}U_{ij//k} = L^{-1}[-m_k\, {}^{1}A_{ij} + n_{(2)k}{}^{3}A_{ij} + U_{(1)k}{}^{5}A_{ij} + 2\, U_{(2)k}{}^{6}A_{ij} + U_{(3)k}{}^{8}A_{ij} + U_{(4)k}{}^{7}A_{ij} - U_{(6)k}{}^{10}A_{ij}], \quad (3.7)b$$

$${}^{3}U_{ij//k} = L^{-1}[-m_k\, {}^{1}A_{ij} + n_{(3)k}{}^{4}A_{ij} + U_{(1)k}{}^{5}A_{ij} + U_{(2)k}{}^{6}A_{ij} + 2\, U_{(4)k}{}^{7}A_{ij} + U_{(5)k}{}^{9}A_{ij} + U_{(6)k}{}^{10}A_{ij}], \quad (3.7)c$$

$${}^{4}U_{ij//k} = L^{-1}[-n_{(1)k}{}^{2}A_{ij} + n_{(2)k}{}^{3}A_{ij} - U_{(1)k}{}^{5}A_{ij} + U_{(2)k}{}^{6}A_{ij} + 2\, U_{(3)k}{}^{8}A_{ij} + U_{(5)k}{}^{9}A_{ij} - U_{(6)k}{}^{10}A_{ij}], \quad (3.7)d$$

$${}^{5}U_{ij//k} = L^{-1}[-n_{(1)k}{}^{2}A_{ij} + n_{(3)k}{}^{4}A_{ij} - U_{(1)k}{}^{5}A_{ij} + U_{(3)k}{}^{8}A_{ij} + U_{(4)k}{}^{7}A_{ij} + 2U_{(5)k}{}^{9}A_{ij} + U_{(6)k}{}^{10}A_{ij}], \quad (3.7)e$$

$${}^{6}U_{ij//k} = L^{-1}[-n_{(2)k}{}^{3}A_{ij} + n_{(3)k}{}^{4}A_{ij} - U_{(2)k}{}^{6}A_{ij} - U_{(3)k}{}^{8}A_{ij} + U_{(4)k}{}^{7}A_{ij} + U_{(5)k}{}^{9}A_{ij} + 2\, U_{(6)k}{}^{10}A_{ij}]. \quad (3.7)f$$

From equations (3.7) a,b,c,d,e,f, we can obtain

$${}^{1}U_{ij//k} + {}^{4}U_{ij//k} = {}^{2}U_{ij//k},\ {}^{1}U_{ij//k} + {}^{5}U_{ij//k} = {}^{2}U_{ij//k} + {}^{6}U_{ij//k} = {}^{3}U_{ij//k},\ {}^{4}U_{ij//k} + {}^{6}U_{ij//k} = {}^{5}U_{ij//k} \quad (3.8)$$

Hence:

Theorem 3.4: In a five-dimensional Finsler space F^5, v-covariant derivatives of the tensor U_{ij} satisfy equation (3.8).

From equation (2.15) a,b,c, we can obtain

$${}^{1}E_{ij//k} = L^{-1}[h_{ik}\, m_j - h_{jk}\, m_i + U_{(1)k}{}^{2}E_{ij} + U_{(2)k}{}^{3}E_{ij} + U_{(4)k}{}^{4}E_{ij}], \quad (3.9)a$$

$${}^{2}E_{ij//k} = L^{-1}[h_{ik}n_{(1)j} - h_{jk}\, n_{(1)I} - U_{(1)k}{}^{1}E_{ij} + U_{(3)k}{}^{3}E_{ij} + U_{(5)k}{}^{4}E_{ij}], \quad (3.9)b$$

$${}^{3}E_{ij//k} = L^{-1}[h_{ik}n_{(2)j} - h_{jk}\, n_{(2)I} - U_{(2)k}{}^{1}E_{ij} - U_{(3)k}{}^{2}E_{ij} + U_{(6)k}{}^{4}E_{ij}], \quad (3.9)c$$

$${}^{4}E_{ij//k} = L^{-1}[h_{ik}n_{(3)j} - h_{jk}\, n_{(3)I} - U_{(4)k}{}^{1}E_{ij} - U_{(5)k}{}^{2}E_{ij} - U_{(6)k}{}^{3}E_{ij}], \quad (3.9)d$$

$${}^{5}E_{ij//k} = L^{-1}[-m_k{}^{2}E_{ij} + n_{(1)k}{}^{1}E_{ij} - U_{(2)k}{}^{8}E_{ij} + U_{(3)k}{}^{6}E_{ij} - U_{(4)k}{}^{9}E_{ij} + U_{(5)k}{}^{7}E_{ij}], \quad (3.9)e$$

$${}^{6}E_{ij//k} = L^{-1}[-m_k{}^{3}E_{ij} + n_{(2)k}{}^{1}E_{ij} + U_{(1)k}{}^{8}E_{ij} - U_{(3)k}{}^{5}E_{ij} - U_{(4)k}{}^{10}E_{ij} - U_{(6)k}{}^{7}E_{ij}], \quad (3.9)f$$

$${}^{7}E_{ij//k} = L^{-1}[-m_k{}^{4}E_{ij} + n_{(3)k}{}^{1}E_{ij} + U_{(1)k}{}^{9}E_{ij} + U_{(2)k}{}^{10}E_{ij} - U_{(5)k}{}^{5}E_{ij} - U_{(6)k}{}^{6}E_{ij}], \quad (3.9)g$$

$${}^{8}E_{ij//k} = L^{-1}[-n_{(1)k}{}^{3}E_{ij} + n_{(2)k}{}^{2}E_{ij} - U_{(1)k}{}^{6}E_{ij} + U_{(2)k}{}^{5}E_{ij} - U_{(5)k}{}^{10}E_{ij} + U_{(6)k}{}^{9}E_{ij}], \quad (3.9)h$$

$${}^{9}E_{ij//k} = L^{-1}[-n_{(2)k}{}^{4}E_{ij} + n_{(3)k}{}^{2}E_{ij} - U_{(2)k}{}^{7}E_{ij} - U_{(3)k}{}^{9}E_{ij} + U_{(4)k}{}^{5}E_{ij} - U_{(6)k}{}^{8}E_{ij}], \quad (3.9)i$$

$${}^{10}E_{ij//k} = L^{-1}[-n_{(2)k}{}^{4}E_{ij} + n_{(3)k}{}^{3}E_{ij} - U_{(2)k}{}^{7}E_{ij} - U_{(3)k}{}^{9}E_{ij} + U_{(4)k}{}^{6}E_{ij} + U_{(5)k}{}^{8}E_{ij}]. \quad (3.9)j$$

From these equations several relations can be established between E-tensors.

D-TENSOR OF FIRST KIND

In a five-dimensional Finsler space F^5, there exist D-tensors of three kinds. Here we shall be defining D-Tensor of first kind. Let ${}^{1}D_{ijk}$ be representing the D-tensor of first kind, which is such that

$${}^{1}D_{ijk}\, l^i = 0 \text{ and } {}^{1}D_{ijk}g^{jk} = {}^{1}D_i = {}^{1}D\, n_{(1)I} \quad (4.1)$$

Any third order tensor in F^5, satisfying equation (4.1) can be expressed as

$${}^{1}D_{ijk} = D_{(1)}\, m_i\, m_j m_k + D_{(2)}\, n_{(1)I}\, n_{(1)j}\, n_{(1)k} + D_{(3)}\, n_{(2)I}\, n_{(2)j}\, n_{(2)k} + D_{(4)}\, n_{(3)I}\, n_{(3)j}\, n_{(3)k}$$

$$+ D_{(5)}\sum_{(ijk)}\{m_i\, m_j\, n_{(1)k}\} + D_{(6)}\sum_{(ijk)}\{m_i\, m_j\, n_{(2)k}\} + D_{(7)}\sum_{(ijk)}\{m_i\, m_j\, n_{(3)k}\}$$

$$+ D_{(8)}\sum_{(ijk)}\{n_{(1)I}\, n_{(1)j} m_k\} + D_{(9)}\sum_{(ijk)}\{n_{(1)I}\, n_{(1)j}\, n_{(2)k}\} + D_{(10)}\sum_{(ijk)}\{n_{(1)I}\, n_{(1)j}\, n_{(3)k}\}$$

$$+ D_{(11)}\sum_{(ijk)}\{n_{(2)I}\, n_{(2)j} m_k\} + D_{(12)}\sum_{(ijk)}\{n_{(2)I}\, n_{(2)j}\, n_{(1)k}\} + D_{(13)}\sum_{(ijk)}\{n_{(2)I}\, n_{(2)j}\, n_{(3)k}\}$$

$$+ D_{(14)}\sum_{(ijk)}\{n_{(3)I}\, n_{(3)j} m_k\} + D_{(15)}\sum_{(ijk)}\{n_{(3)I}\, n_{(3)j}\, n_{(1)k}\} + D_{(16)}\sum_{(ijk)}\{n_{(3)I}\, n_{(3)j}\, n_{(2)k}\}$$

$$+ D_{(17)}\sum_{(ijk)}\{m_i(n_{(1)j}\, n_{(2)k} + n_{(1)k}\, n_{(2)j})\} + D_{(18)}\sum_{(ijk)}\{m_i(n_{(2)j}\, n_{(3)k} + n_{(2)k}\, n_{(3)j})\}$$

$$+ D_{(19)}\sum_{(ijk)}\{m_i\,(n_{(3)j}\, n_{(1)k} + n_{(3)k}\, n_{(1)j})\} + D_{(20)}\sum_{(ijk)}\{n_{(1)i}(n_{(2)j}\, n_{(3)k} + n_{(2)k}\, n_{(3)j})\} \quad (4.2)$$

Definition 4.1: In a five-dimensional Finsler space F^5, the tensor ${}^1D_{ijk}$, defined by equation (4.2) is called D-tensor of first kind.

Multiplying equation (4.2) by g^{jk}, we obtain on simplification

$$^1D_i = m_i(D_{(1)} + D_{(8)} + D_{(11)} + D_{(14)}) + n_{(1)I}\,(D_{(2)} + D_{(5)} + D_{(12)} + D_{(15)}) + n_{(2)i}(D_{(3)} + D_{(6)} + D_{(9)} + D_{(16)})$$

$$+ n_{(3)i}(D_{(4)} + D_{(7)} + D_{(10)} + D_{(13)}), \quad (4.3)$$

which by virtue of (4.1) leads to

$$D_{(1)} + D_{(8)} + D_{(11)} + D_{(14)} = 0,\ D_{(2)} + D_{(5)} + D_{(12)} + D_{(15)} = {}^1D,\ D_{(3)} + D_{(6)} + D_{(9)} + D_{(16)} = 0,$$

$$D_{(4)} + D_{(7)} + D_{(10)} + D_{(13)} = 0. \quad (4.4)$$

Hence:

Theorem 4.1: In a five-dimensional Finsler space F^5, the 16 coefficients of the tensor ${}^1D_{ijk}$, defined by equation (4.2) satisfy equation (4.4).

Let us assume that the tensor ${}^1D_{ijk} = 0$, then from equation (4.2) with the help of (4.4), we observe that

$$D_{(2)} + D_{(5)} + D_{(12)} + D_{(15)} = 0, \quad (4.5)$$

which with the help of equation (4.3) leads to

Theorem 4.2: In a five-dimensional Finsler space F^5, the necessary and sufficient condition for the vector ${}^1D_i$ to vanish is given by equation (4.5).

Equation (4.2) can alternatively be expressed as

$$^1D_{ijk} = \sum_{(ijk)}\{m_i W_{jk} + n_{(1)I} X_{jk} + n_{(2)I} Y_{jk} + n_{(3)I} Z_{jk}\}, \quad (4.6)$$

Where

$$W_{jk} = (1/3)[D_{(1)}\, m_j m_k + 3\, D_{(8)}\, n_{(1)j}\, n_{(1)k} + 3\, D_{(11)}\, n_{(2)j}\, n_{(2)k} + 3\, D_{(14)}\, n_{(3)j}\, n_{(3)k}$$

$$+ D_{(17)}(n_{(1)j}\, n_{(2)k} + n_{(1)k}\, n_{(2)j}) + D_{(18)}(n_{(2)j}\, n_{(3)k} + n_{(2)k}\, n_{(3)j}) + D_{(19)}(n_{(3)j}\, n_{(1)k} + n_{(3)k}\, n_{(1)j})], \quad (4.7)a$$

$$X_{jk} = (1/3)[D_{(2)}\, n_{(1)j}\, n_{(1)k} + 3\, D_{(5)} m_j m_k + 3\, D_{(12)}\, n_{(2)j}\, n_{(2)k} + 3\, D_{(15)}\, n_{(3)j}\, n_{(3)k}$$

$$+ D_{(17)}(m_j\, n_{(2)k} + m_k\, n_{(2)j}) + D_{(19)}(n_{(3)j} m_k + n_{(3)k}\, m_j) + D_{(20)}(n_{(2)j}\, n_{(3)k} + n_{(2)k}\, n_{(3)j})], \quad (4.7)b$$

$$Y_{jk} = (1/3)[D_{(3)}\, n_{(2)j}\, n_{(2)k} + 3\, D_{(6)}\, m_j m_k + 3\, D_{(9)}\, n_{(1)j}\, n_{(1)k} + 3\, D_{(16)} n_{(3)j}\, n_{(3)k}$$

$$+ D_{(17)}(m_j\, n_{(1)k} + m_k\, n_{(1)j}) + D_{(18)}(n_{(3)j}m_k + n_{(3)k}m_j) + D_{(20)}(n_{(1)k}\, n_{(3)j} + n_{(1)j}\, n_{(3)k})], \qquad (4.7)c$$

$$Z_{jk} = (1/3)[D_{(4)}\, n_{(3)j}\, n_{(3)k} + 3\, D_{(7)}\, m_j m_k + 3\, D_{(10)}\, n_{(1)j}\, n_{(1)k} + 3\, D_{(13)}\, n_{(2)j}\, n_{(2)k}$$

$$+ D_{(18)}(m_j\, n_{(2)k} + m_k\, n_{(2)j}) + D_{(19)}(m_j\, n_{(1)k} + m_k\, n_{(1)j}) + D_{(20)}(n_{(1)j}\, n_{(2)k} + n_{(1)k}\, n_{(2)j})] \qquad (4.7)d$$

Multiplying equation (4.2) respectively by m^k, $n_{(1)}{}^k$, $n_{(2)}{}^k$and $n_{(3)}{}^k$ and using

$$^1D_{ij} = {}^1D_{ijk}m^k,\ {}^{11}D_{ij} = {}^1D_{ijk}n_{(1)}{}^k,\ {}^{12}D_{ij} = {}^1D_{ijk}\, n_{(2)}{}^k \text{ and } {}^{13}D_{ij} = {}^1D_{ijk}\, n_{(3)}{}^k \qquad (4.8)$$

together with equations (2.1), (2.6) and (2.9), we get on simplification similar to Shimada [17]

$$^1D_{ij} = D_{(1)}\,{}^1B_{ij} + D_{(8)}\,{}^2B_{ij} + D_{(11)}\,{}^3B_{ij} + D_{(14)}{}^4B_{ij} + D_{(5)}{}^5A_{ij} + D_{(6)}\,{}^6A_{ij}$$

$$+ D_{(7)}{}^7A_{ij} + D_{(17)}{}^8A_{ij} + D_{(18)}{}^{10}A_{ij} + D_{(19)}{}^9A_{ij}, \qquad (4.9)a$$

$$^{11}D_{ij} = D_{(2)}{}^2B_{ij} + D_{(5)}{}^1B_{ij} + D_{(12)}{}^3B_{ij} + D_{(15)}{}^4B_{ij} + D_{(8)}{}^5A_{ij} + D_{(9)}{}^8A_{ij}$$

$$+ D_{(10)}{}^9A_{ij} + D_{(17)}{}^6A_{ij} + D_{(19)}{}^7A_{ij} + D_{(20)}{}^{10}A_{ij}, \qquad (4.9)b$$

$$^{12}D_{ij} = D_{(3)}{}^3B_{ij} + D_{(6)}{}^1B_{ij} + D_{(9)}{}^2B_{ij} + D_{(16)}{}^4B_{ij} + D_{(11)}{}^6A_{ij} + D_{(12)}{}^8A_{ij}$$

$$+ D_{(13)}{}^{10}A_{ij} + D_{(17)}{}^5A_{ij} + D_{(18)}{}^7A_{ij} + D_{(20)}{}^9A_{ij}, \qquad (4.9)c$$

$$^{13}D_{ij} = D_{(4)}\,{}^4B_{ij} + D_{(7)}{}^1B_{ij} + D_{(10)}{}^2B_{ij} + D_{(13)}{}^3B_{ij} + D_{(14)}{}^7A_{ij} + D_{(15)}{}^9A_{ij}$$

$$+ D_{(16)}\,{}^{10}A_{ij} + D_{(18)}\,{}^6A_{ij} + D_{(19)}{}^5A_{ij} + D_{(20)}{}^8A_{ij}. \qquad (4.9)d$$

From equation (4.9)a,b,c,d, it is easy to observe that

$$^1D_{ijk} = {}^1D_{ij}m_k + {}^{11}D_{ij}n_{(1)k} + {}^{12}D_{ij}\, n_{(2)k} + {}^{13}D_{ij}\, n_{(3)k} \qquad (4.10)$$

From equations (4.9)a,b,c,d, we can easily obtain

$$^1D_{ij}\, m^j = D_{(1)}\, m_i + D_{(5)}\, n_{(1)I} + D_{(6)}\, n_{(2)I} + D_{(7)}\, n_{(3)I}, \qquad (4.11)a$$

$$^{11}D_{ij}n_{(1)}{}^j = D_{(2)}\, n_{(1)I} + D_{(8)}\, m_i + D_{(9)}\, n_{(2)I} + D_{(10)}\, n_{(3)I}, \qquad (4.11)b$$

$$^{12}D_{ij}n_{(2)}{}^j = D_{(3)}\, n_{(2)I} + D_{(11)}\, m_i + D_{(12)}\, n_{(1)I} + D_{(13)}\, n_{(3)I}, \qquad (4.11)c$$

$$^{13}D_{ij}n_{(3)}{}^j = D_{(4)}\, n_{(3)I} + D_{(14)}\, m_i + D_{(15)}\, n_{(1)I} + D_{(16)}\, n_{(2)I}, \qquad (4.11)d$$

Adding all these equations and using equation (4.4), we get

$$^1D_{ij}\, m^j + {}^{11}D_{ij}n_{(1)}{}^j + {}^{12}D_{ij}\, n_{(2)}{}^j + {}^{13}D_{ij}\, n_{(3)}{}^j = {}^1D_i \qquad (4.12)$$

Hence:

Theorem 4.3: The vector ^1D_i in a five-dimensional Finsler space F^5, satisfies equation (4.12).

The h-covariant derivative of tensor $^1D_{ijk}$ can be obtained as

$$^1D_{ijk/h} = A_{(1)h}\, m_i\, m_j m_k + A_{(2)h}\, n_{(1)i}\, n_{(1)j}\, n_{(1)k} + A_{(3)h}\, n_{(2)i}n_{(2)j}\, n_{(2)k} + A_{(4)h}\, n_{(3)I}\, n_{(3)j}\, n_{(3)k}$$

$$+ \sum_{(I,j,k)} [A_{(5)h}\, \{m_i\, m_j\, n_{(1)k}\} + A_{(6)h}\, \{ m_i\, m_j\, n_{(2)k}\} + A_{(7)h}\, \{ m_i\, m_j\, n_{(3)k}\}$$

$$+ A_{(8)h}\, \{ n_{(1)i}\, n_{(1)jj}m_k\} + A_{(9)h}\, \{ n_{(1)i}\, n_{(1)j}n_{(2)k}\} + A_{(10)h}\, \{n_{(1)i}\, n_{(1)j}n_{(3)k}\}$$

$+ A_{(11)h} \{n_{(2)i}n_{(2)j}m_k\} + A_{(12)h} \{n_{(2)i}n_{(2)j}\, n_{(1)k}\} + A_{(13)h} \{n_{(2)i}n_{(2)j}n_{(3)k}\}$

$+ A_{(14)h} \{n_{(3)i}n_{(3)j}m_k\} + A_{(15)h} \{n_{(3)i}n_{(3)j}\, n_{(1)k}\} + A_{(16)h} \{ n_{(3)i}n_{(3)j}n_{(2)k}\}$

$+ A_{(17)h} \{m_i(n_{(1)j}n_{(2)k} + n_{(1)k}n_{(2)j})\} + A_{(18)h}\{m_i(n_{(1j}n_{(3)k} + n_{(1)k}n_{(3)j})\}$

$+ A_{(19)h} \{m_i(n_{(2)j}n_{(3)k} + n_{(2)k}n_{(3)j})\} + A_{(20)h} \{n_{(1)i}(n_{(2)j}n_{(3)k} + n_{(2)k}n_{(3)j})\}$ (4.13)

where we have used

$A_{(1)j} = D_{(1)/j} + 3(D_{(6)}h_{(3)j} – D_{(5)}\, h_{(1)j} + D_{(7)}h_{(4)j})$

$A_{(2)j} = D_{(2)/j} + 3(D_{(8)}\, h_{(1)j} – D_{(9)}h_{(2)j} + D_{(10)}h_{(5)j})$

A(3)j = D(3)/j + 3(D(12)h(2)j – D(11)h(3)j + D(13)h(6)j)

$A_{(4)j} = D_{(4)/j} – 3(D_{(14)}h_{(4)j} + D_{(15)}h_{(5)j} + D_{(16)}h_{(6)j})$

$A_{(5)j} = D_{(5)/j} + (D_{(1)} – 2D_{(8)})h_{(1)j} – D_{(6)}h_{(2)j} + D_{(7)}h_{(5)j} + 2\, D_{(17)}h_{(3)j} + 2D_{(18)}h_{(4)j}$

$A_{(6)j} = D_{(6)/j} – (D_{(1)} – 2\, D_{(11)})h_{(3)j} + D_{(5)}h_{(2)j} + D_{(7)}h_{(6)j} – 2\, D_{(17)}\, h_{(1)j} + 2\, D_{(19)}h_{(4)}$

$A_{(7)j} = D_{(7)/j} -(D_{(1)} – 2\, D_{(14)})h_{(4)j} – D_{(5)}h_{(5)j} – D_{(6)}h_{(6)j} – 2\, D_{(18)}\, h_{(1)j} + 2\, D_{(19)}h_{(3)}$

$A_{(8)j} = D_{(8)/j} – (D_{(2)} – 2\, D_{(5)})h_{(1)j} + D_{(9)}h_{(3)j} + D_{(10)}h_{(4)j} – 2\, D_{(17)}\, h_{(2)j} + 2\, D_{(18)}h_{(5)j}$

$A_{(9)j} = D_{(9)/j} + (D_{(2)} – 2D_{(12)})\, h_{(2)j} – D_{(8)}h_{(3)j} + D_{(10)}h_{(6)j} + 2\, D_{(17)}\, h_{(1)j} + 2\, D_{(20)}h_{(5)j}$

$A_{(10)j} = D_{(10)/j} - (D_{(2)} – 2\, D_{(15)})h_{(5)j} – D_{(8)}h_{(4)j} - D_{(9)}h_{(6)j} + 2\, d_{(18)}\, h_{(1)j} – 2\, D_{(20)}h_{(2)j}$

$A_{(11)j} = D_{(11)/j} + (D_{(3)} - 2\, D_{(6)})h_{(3)j} - D_{(12)}\, h_{(1)j} + D_{(13)}h_{(4)j} + 2D_{(17)}h_{(2)j} + 2D_{(19)}h_{(6)j}$

$A_{(12)j} = D_{(12)/j} + D_{(11)}\, h_{(1)j} -(D_{(3)} - 2\, D_{(9)})h_{(2)j} + D_{(13)}h_{(5)j} – 2\, D_{(17)}h_{(3)j} + 2\, D_{(20)}h_{(6)j}$

$A_{(13)j} = D_{(13)/j} -(D_{(3)} – 2\, D_{(16)})\, h_{(6)j} – D_{(11)}h_{(4)j} – D_{(12)}h_{(5)j} – 2\, D_{(19)}h_{(3)j} + 2\, D_{(20)}h_{(2)j}$

$A_{(14)j} = D_{(14)/j} + (D_{(4)} – 2\, D_{(7)})h_{(4)j} – D_{(15)}\, h_{(1)j} + D_{(16)}h_{(3)j} – 2\, D_{(18)}h_{(5)j} -2\, D_{(19)}h_{(6)j}$

$A_{(15)j} = D_{(15)/j} + (D_{(4)} – 2\, D_{(10)})\, h_{(5)j} + D_{(14)}\, h_{(1)j} – D_{(16)}h_{(2)j} -2\, D_{(18)}h_{(4)j} – 2\, D_{(20)}h_{(6)j}$

$A_{(16)j} = D_{(16)/j} + (D_{(4)} – 2\, D_{(13)})h_{(6)j} – D_{(14)}h_{(3)j} + D_{(15)}h_{(2)j} – 2\, D_{(19)}h_{(4)j} – 2\, D_{(20)}h_{(5)j}$

$A_{(17)j} = D_{(17)/j} – D_{(5)}h_{(3)j} + (D_{(8)} – D_{(11)})h_{(2)j} + (D_{(6)} – D_{(9)})h_{(1)j} + D_{(12)}h_{(3)j} + D_{(18)}h_{(6)j}$

$+ D_{(19)}h_{(5)j} + D_{(20)}h_{(4)j}$

$A_{(18)j} = D_{(18)/j} – (D_{(5)} – D_{(15)})h_{(4)j} – (D_{(8)} – D_{(14)})\, h_{(5)j} – D_{(17)}h_{(6)j} + (D_{(7)} – D_{(10)})h_{(1)j}$

$– D_{(19)}h_{(2)j} + D_{(20)}h_{(3)j}$

$A_{(19)j} = D_{(19)/j} – D_{(17)}h_{(5)j} -(D_{(7)} – D_{(13)})h_{(3)j} – (D_{(6)} – D_{(16)})h_{(4)j} – (D_{(11)} – D_{(14)})h_{(6)j}$

$+ D_{(18)}h_{(2)j} – D_{(20)}\, h_{(1)j}$

$A_{(20)j} = D_{(20)/j} + (D_{(10)} – D_{(13)})\, h_{(2)j} – (D_{(9)} – D_{(16)})h_{(5)j} – D_{(17)}h_{(4)j} -(D_{(12)} – D_{(15)})\, h_{(6)j}$

$– D_{(18)}h_{(3)j} + D_{(19)}\, h_{(1)j}$ (4.14)

From equation (4.13), we can obtain by virtue of ${}^1D_{ijk/h}l^h = {}^1D_{ijk/0}$, similar to Izumi [5]

$$
\begin{aligned}
{}^1D_{ijk/0} = {} & A_{(1)0}\, m_i\, m_j m_k + A_{(2)0}\, n_{(1)i}\, n_{(1)j}\, n_{(1)k} + A_{(3)0}\, n_{(2)i} n_{(2)j}\, n_{(2)k} + A_{(4)0}\, n_{(3)I}\, n_{(3)j}\, n_{(3)k} \\
& + \textstyle\sum_{(I,j,k)} [A_{(5)0}\{m_i\, m_j\, n_{(1)k}\} + A_{(6)0}\, \{ m_i\, m_j\, n_{(2)k}\} + A_{(7)0}\, \{ m_i\, m_j\, n_{(3)k}\} \\
& + A_{(8)0}\, \{ n_{(1)i}\, n_{(1)jj} m_k\} + A_{(9)0}\, \{ n_{(1)i}\, n_{(1)j}\, n_{(2)k}\} + A_{(10)0}\, \{n_{(1)i}\, n_{(1)j}\, n_{(3)k}\} \\
& + A_{(11)0}\{n_{(2)i}\, n_{(2)j} m_k\} + A_{(12)0}\, \{n_{(2)i}\, n_{(2)j}\, n_{(1)k}\} + A_{(13)0}\, \{n_{(2)i}\, n_{(2)j}\, n_{(3)k}\} \\
& + A_{(14)0}\, \{n_{(3)i}\, n_{(3)j} m_k\} + A_{(15)0}\, \{n_{(3)i}\, n_{(3)j}\, n_{(1)k}\} + A_{(16)0}\, \{n_{(3)i}\, n_{(3)j}\, n_{(2)k}\} \\
& + A_{(17)0}\, \{m_i(n_{(1)j}\, n_{(2)k} + n_{(1)k}\, n_{(2)j})\} + A_{(18)0}\{m_i(n_{(1j}\, n_{(3)k} + n_{(1)k}\, n_{(3)j})\} \\
& + A_{(19)0}\, \{m_i(n_{(2)j}\, n_{(3)k} + n_{(2)k}\, n_{(3)j})\} + A_{(20)0}\, \{n_{(1)i}(n_{(2)j}\, n_{(3)k} + n_{(2)k}\, n_{(3)j})\}
\end{aligned} \tag{4.15}
$$

If we assume that in a Finsler space of five-dimensions tensor ${}^1D_{ijk/0} = \lambda {}^1D_{ijk}$, from equations (4.12) and (4.15) we get $A_{(r)0} = \lambda D_{(r)}$, (r = 1,…,20). Hence:

Theorem 4.4: In a Finsler space of five-dimensions, tensor ${}^1D_{ijk}$ satisfies ${}^1D_{ijk/0} = \lambda\, {}^1D_{ijk}$ if and only if coefficients of these tensors satisfy $A_{(r)0} = \lambda D_{(r)}$, (r = 1,…,20).

D-TENSOR OF SECOND KIND

In this section we shall define a symmetric tensor of second kind, which shall be denoted by ${}^2D_{ijk}$ and which satisfies ${}^2D_{ijk}\, l^i = 0$ as well as ${}^2D_{ijk} g^{jk} = {}^2D_i = {}^2D n_{(2)i}$. Any third order tensor satisfying these properties in a Finsler space of five-dimensions will be expressed as

$$
\begin{aligned}
{}^2D_{ijk} = {} & {}^*D_{(1)}\, m_i\, m_j m_k + {}^*D_{(2)}\, n_{(1)I}\, n_{(1)j}\, n_{(1)k} + {}^*D_{(3)}\, n_{(2)I}\, n_{(2)j}\, n_{(2)k} + {}^*D_{(4)}\, n_{(3)I}\, n_{(3)j}\, n_{(3)k} \\
& + {}^*D_{(5)}\textstyle\sum_{(ijk)}\{m_i\, m_j\, n_{(1)k}\} + {}^*D_{(6)}\sum_{(ijk)}\{m_i\, m_j\, n_{(2)k}\} + {}^*D_{(7)}\sum_{(ijk)}\{m_i\, m_j\, n_{(3)k}\} \\
& + {}^*D_{(8)}\textstyle\sum_{(ijk)}\{n_{(1)I}\, n_{(1)j} m_k\} + {}^*D_{(9)}\sum_{(ijk)}\{n_{(1)I}\, n_{(1)j}\, n_{(2)k}\} + {}^*D_{(10)}\sum_{(ijk)}\{n_{(1)I}\, n_{(1)j}\, n_{(3)k}\} \\
& + {}^*D_{(11)}\textstyle\sum_{(ijk)}\{n_{(2)I}\, n_{(2)j} m_k\} + {}^*D_{(12)}\sum_{(ijk)}\, \{n_{(2)I}\, n_{(2)j}\, n_{(1)k}\} + {}^*D_{(13)}\sum_{(ijk)}\{n_{(2)I}\, n_{(2)j}\, n_{(3)k}\} \\
& + {}^*D_{(14)}\textstyle\sum_{(ijk)}\{n_{(3)I}\, n_{(3)j} m_k\} + {}^*D_{(15)}\sum_{(ijk)}\{n_{(3)I}\, n_{(3)j}\, n_{(1)k}\} + {}^*D_{(16)}\sum_{(ijk)}\{n_{(3)I}\, n_{(3)j}\, n_{(2)k}\} \\
& + {}^*D_{(17)}\textstyle\sum_{(ijk)}\{m_i(n_{(1)j}\, n_{(2)k} + n_{(1)k}\, n_{(2)j})\} + {}^*D_{(18)}\sum_{(ijk)}\{m_i(n_{(2)j}\, n_{(3)k} + n_{(2)k}\, n_{(3)j})\} \\
& + {}^*D_{(19)}\textstyle\sum_{(ijk)}\{m_i\, (n_{(3)j}\, n_{(1)k} + n_{(3)k}\, n_{(1)j})\} + {}^*D_{(20)}\sum_{(ijk)}\{n_{(1)i}(n_{(2)j}\, n_{(3)k} + n_{(2)k}\, n_{(3)j})\}
\end{aligned} \tag{5.1}
$$

Multiplying equation (5.1) by g^{jk}, we obtain on simplification

$$
\begin{aligned}
{}^2D_i = {} & m_i({}^*D_{(1)} + {}^*D_{(8)} + {}^*D_{(11)} + {}^*D_{(14)}) + n_{(1)I}\, ({}^*D_{(2)} + {}^*D_{(5)} + {}^*D_{(12)} + {}^*D_{(15)}) \\
& + n_{(2)i}({}^*D_{(3)} + {}^*D_{(6)} + {}^*D_{(9)} + {}^*D_{(16)}) + n_{(3)i}({}^*D_{(4)} + {}^*D_{(7)} + {}^*D_{(10)} + {}^*D_{(13)})
\end{aligned} \tag{5.2}
$$

Now using ${}^2D_i = {}^2D n_{(2)I}$, in equation (5.2), we get

$$ {}^*D_{(1)} + {}^*D_{(8)} + {}^*D_{(11)} + {}^*D_{(14)} = 0,\ {}^*D_{(2)} + {}^*D_{(5)} + {}^*D_{(12)} + {}^*D_{(15)} = 0 \tag{5.3a} $$

$$ {}^*D_{(3)} + {}^*D_{(6)} + {}^*D_{(9)} + {}^*D_{(16)} = {}^2D,\ {}^*D_{(4)} + {}^*D_{(7)} + {}^*D_{(10)} + {}^*D_{(13)} = 0 \tag{5.3b} $$

Hence

Theorem 5.1: In a five-dimensional Finsler space F^5, D-tensor of second kind denoted by $^2D_{ijk}$ and given by equation (5.1) satisfies equations (5.3)a,b.

If we assume that tensor $^2D_{ijk} = 0$, we can observe that this will also satisfy equation

$$^*D_{(3)} + {}^*D_{(6)} + {}^*D_{(9)} + {}^*D_{(16)} = 0. \quad (5.4)$$

Hence:

Theorem 5.2: In a five-dimensional Finsler space F^5, if the tensor $^2D_{ijk}$ vanishes equation (5.4) is satisfied.

Alternatively, this tensor can also be expressed as

$$^2D_{ijk} = \sum_{(ijk)}[m_i{}^*W_{jk} + n_{(1)I}{}^*X_{jk} + n_{(2)I}{}^*Y_{jk} + n_{(3)I}{}^*Z_{jk}], \quad (5.5)$$

where

$$^*W_{jk} = (1/3)[{}^*D_{(1)}\, m_j m_k + 3\,{}^*D_{(8)}\, n_{(1)j}\, n_{(1)k} + 3\,{}^*D_{(11)}\, n_{(2)j}\, n_{(2)k} + 3\,{}^*D_{(14)}\, n_{(3)j}\, n_{(3)k}$$
$$+ {}^*D_{(17)}(n_{(1)j}\, n_{(2)k} + n_{(1)k}\, n_{(2)j}) + {}^*D_{(18)}(n_{(2)j}\, n_{(3)k} + n_{(2)k}\, n_{(3)j}) + {}^*D_{(19)}(n_{(3)j}\, n_{(1)k} + n_{(3)k}\, n_{(1)j})], \quad (5.6)a$$

$$^*X_{jk} = (1/3)[{}^*D_{(2)}\, n_{(1)j}\, n_{(1)k} + 3\,{}^*D_{(5)}\, m_j m_k + 3\,{}^*D_{(12)}\, n_{(2)j}\, n_{(2)k} + 3\,{}^*D_{(15)}\, n_{(3)j}\, n_{(3)k}$$
$$+ {}^*D_{(17)}(m_j\, n_{(2)k} + m_k\, n_{(2)j}) + {}^*D_{(19)}(n_{(3)j} m_k + n_{(3)k}\, m_j) + {}^*D_{(20)}(n_{(2)j}\, n_{(3)k} + n_{(2)k}\, n_{(3)j})], \quad (5.6)b$$

$$^*Y_{jk} = (1/3)[{}^*D_{(3)}\, n_{(2)j}\, n_{(2)k} + 3\,{}^*D_{(6)}\, m_j m_k + 3\,{}^*D_{(9)}\, n_{(1)j}\, n_{(1)k} + 3\,{}^*D_{(16)} n_{(3)j}\, n_{(3)k}$$
$$+ {}^*D_{(17)}(m_j\, n_{(1)k} + m_k\, n_{(1)j}) + {}^*D_{(18)}(n_{(3)j} m_k + n_{(3)k}\, m_j) + {}^*D_{(20)}(n_{(1)k}\, n_{(3)j} + n_{(1)j}\, n_{(3)k})], \quad (5.6)c$$

$$^*Z_{jk} = (1/3)[{}^*D_{(4)}\, n_{(3)j}\, n_{(3)k} + 3\,{}^*D_{(7)}\, m_j m_k + 3\,{}^*D_{(10)}\, n_{(1)j}\, n_{(1)k} + 3\,{}^*D_{(13)}\, n_{(2)j}\, n_{(2)k}$$
$$+ {}^*D_{(18)}(m_j\, n_{(2)k} + m_k\, n_{(2)j}) + {}^*D_{(19)}(m_j\, n_{(1)k} + m_k\, n_{(1)j}) + {}^*D_{(20)}(n_{(1)j}\, n_{(2)k} + n_{(1)k}\, n_{(2)j})] \quad (5.6)d$$

D-TENSOR OF THIRD KIND

In this section, we shall define a symmetric tensor of third kind, which shall be denoted by $^3D_{ijk}$ and which satisfies $^3D_{ijk}\, l^i = 0$ as well as $^3D_{ijk} g^{jk} = {}^3D_i = {}^3D n_{(3)i}$. Any third order tensor satisfying these properties in a Finsler space of five-dimensions will be expressed as

$$^3D_{ijk} = {}'D_{(1)}\, m_i\, m_j m_k + {}'D_{(2)}\, n_{(1)I}\, n_{(1)j}\, n_{(1)k} + {}'D_{(3)}\, n_{(2)I}\, n_{(2)j}\, n_{(2)k} + {}'D_{(4)}\, n_{(3)I}\, n_{(3)j}\, n_{(3)k}$$
$$+ {}'D_{(5)}\sum_{(ijk)}\{m_i\, m_j\, n_{(1)k}\} + {}'D_{(6)}\sum_{(ijk)}\{m_i\, m_j\, n_{(2)k}\} + {}'D_{(7)}\sum_{(ijk)}\{m_i\, m_j\, n_{(3)k}\}$$
$$+ {}'D_{(8)}\sum_{(ijk)}\{n_{(1)I}\, n_{(1)j} m_k\} + {}'D_{(9)}\sum_{(ijk)}\{n_{(1)I}\, n_{(1)j}\, n_{(2)k}\} + {}'D_{(10)}\sum_{(ijk)}\{n_{(1)I}\, n_{(1)j}\, n_{(3)k}\}$$
$$+ {}'D_{(11)}\sum_{(ijk)}\{n_{(2)I}\, n_{(2)j} m_k\} + {}'D_{(12)}\sum_{(ijk)}\{n_{(2)I}\, n_{(2)j}\, n_{(1)k}\} + {}'D_{(13)}\sum_{(ijk)}\{n_{(2)I}\, n_{(2)j}\, n_{(3)k}\}$$
$$+ {}'D_{(14)}\sum_{(ijk)}\{n_{(3)I}\, n_{(3)j} m_k\} + {}'D_{(15)}\sum_{(ijk)}\{n_{(3)I}\, n_{(3)j}\, n_{(1)k}\} + {}'D_{(16)}\sum_{(ijk)}\{n_{(3)I}\, n_{(3)j}\, n_{(2)k}\}$$
$$+ {}'D_{(17)}\sum_{(ijk)}\{m_i(n_{(1)j}\, n_{(2)k} + n_{(1)k}\, n_{(2)j})\} + {}'D_{(18)}\sum_{(ijk)}\{m_i(n_{(2)j}\, n_{(3)k} + n_{(2)k}\, n_{(3)j})\}$$
$$+ {}'D_{(19)}\sum_{(ijk)}\{m_i\,(n_{(3)j}\, n_{(1)k} + n_{(3)k}\, n_{(1)j})\} + {}'D_{(20)}\sum_{(ijk)}\{n_{(1)i}(n_{(2)j}\, n_{(3)k} + n_{(2)k}\, n_{(3)j})\} \quad (6.1)$$

From equation (6.1), we can obtain

$$^3D_i = m_i({}'D_{(1)} + {}'D_{(8)} + {}'D_{(11)} + {}'D_{(14)}) + n_{(1)I}\,({}'D_{(2)} + {}'D_{(5)} + {}'D_{(12)} + {}'D_{(15)})$$

$$+ n_{(2)i}({}'D_{(3)} + {}'D_{(6)} + {}'D_{(9)} + {}'D_{(16)}) + n_{(3)i}({}'D_{(4)} + {}'D_{(7)} + {}'D_{(10)} + {}'D_{(13)}), \quad (6.2)$$

which implies

$${}'D_{(1)} + {}'D_{(8)} + {}'D_{(11)} + {}'D_{(14)} = 0,\ {}'D_{(2)} + {}'D_{(5)} + {}'D_{(12)} + {}'D_{(15)} = 0, \quad (6.3)a$$

$${}'D_{(3)} + {}'D_{(6)} + {}'D_{(9)} + {}'D_{(16)} = 0,\ {}'D_{(4)} + {}'D_{(7)} + {}'D_{(10)} + {}'D_{(13)} = {}^3D \quad (6.3)b$$

Hence:

Theorem 6.1: In a five-dimensional Finsler space F^5, the coefficients on the right-hand side of ${}^3D_{ijk}$ satisfy equations (6.3)a,b.

If we assume that tensor ${}^3D_{ijk} = 0$, equation (6.3) b implies

$${}'D_{(4)} + {}'D_{(7)} + {}'D_{(10)} + {}'D_{(13)} = 0. \quad (6.4)$$

Hence:

Theorem 6.2: In a five- dimensional Finsler space F^5, if the tensor ${}^3D_{ijk}$ vanishes, equation (6.4) is satisfied.

Remarks

- Tensors ${}^2D_{ijk}$ and ${}^3D_{ijk}$ also satisfy properties similar to ${}^1D_{ijk}$.
- Curvature properties related with these tensors are being studied in the subsequent research work.

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